

Normal Constraint Method with Guarantee of Even Representation of Complete Pareto Frontier

Achille Messac* and Christopher A. Mattson†
Rensselaer Polytechnic Institute, Troy, New York 12180

Multiobjective optimization is rapidly becoming an invaluable tool in engineering design. A particular class of solutions to the multiobjective optimization problem is said to belong to the Pareto frontier. A Pareto solution, the set of which comprises the Pareto frontier, is optimal in the sense that any improvement in one design objective can only occur with the worsening of at least one other. Accordingly, the Pareto frontier plays an important role in engineering design—it characterizes the tradeoffs between conflicting design objectives. Some optimization methods can be used to automatically generate a set of Pareto solutions from which a final design is subjectively chosen by the designer. For this approach to be successful, the generated Pareto set must be truly representative of the complete optimal design space (Pareto frontier). In other words, the set must not overrepresent one region of the design space, or neglect others. Some commonly used methods comply with this requirement, whereas others do not. This paper offers a new phase in the development of the normal constraint method, which is a simple approach for generating Pareto solutions that are evenly distributed in the design space of an arbitrary number of objectives. The even distribution of the generated Pareto solutions can facilitate the process of developing an analytical expression for the Pareto frontier in n dimension. An even distribution of Pareto solutions also facilitates the task of choosing the most desirable (final) design from among the set of Pareto solutions. The normal constraint method bears some similarities to the normal boundary intersection and ϵ -constraint methods. Importantly, the developments presented in this paper define its critical distinction, namely, the ability to generate a set of evenly distributed Pareto solutions over the complete Pareto frontier. Examples are provided that show the normal constraint method to perform favorably under the new developments when compared with the normal boundary intersection method, as well as with the original normal constraint method.

Nomenclature

g	=	vector of inequality constraints
h	=	vector of equality constraints
m	=	number of design objectives
n_p	=	number of generated Pareto points
p	=	vector of points on the utopia plane
v_i	=	vector from anchor points i to j ($i \neq j$)
x	=	vector of design variables
x^{i*}	=	x corresponding to μ^{i*}
α	=	nondimensional parameter
δ^i	=	i th fixed increment
μ	=	vector of design metrics (objectives)
μ^{i*}	=	i th anchor point

Subscripts and Superscripts

i, j, k, r	=	dummy indices
l	=	minimal value or lower bound
N	=	indicates nadir (worst)
U	=	indicates utopia (ideal)
u	=	maximum value or upper bound
$(\bar{})$	=	normalized form of variable ()
$*$	=	indicates optimum

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*Professor, Department of Mechanical, Aerospace, and Nuclear Engineering; messac@rpi.edu. Associate Fellow AIAA.

†Ph.D. Candidate, Department of Mechanical, Aerospace, and Nuclear Engineering. Student Member AIAA.

I. Introduction

MULTIOBJECTIVE optimization and multiattribute utility theory are important technical areas that have made significant contributions to engineering design theory and practice. Different segments of the design community hold somewhat differing views of the relative or absolute merits of the associated methods for credible reasons, yet others advocate the use of a single objective (e.g., profit), instead of multiple conflicting objectives. This paper takes the view that in any engineering design the designer will generally need to balance different practical concerns. Accordingly, he or she will find it useful to obtain a solution that explicitly or implicitly addresses conflicting design objectives.

A. Pareto Optimality in Engineering Design

The concept of Pareto optimality has played an important role in the process of identifying solutions that explicitly recognize the conflicting nature of optimal solutions, in terms of balancing competing objectives. Specifically, a Pareto solution is one where any improvement in one objective can only occur through the worsening of at least one other objective.^{1–3} The fundamental nature of this property makes Pareto solutions critical to optimal engineering design. When one chooses a design that is not Pareto optimal, one essentially forfeits improvements that would otherwise entail no compromise. In other words, one is giving up something that is free. Only rarely can practical reasons exist for doing so.

In the process of engineering design optimization, one basically has two options for identifying Pareto solutions. The first is to use an aggregate objective function (AOF), which is optimized (minimized in this paper) subject to behavioral and side constraints to yield the most preferred design. Unfortunately, the initial AOF generally fails to reflect the designer's true (and complex) preference. This initial failure to obtain the most preferred design is generally followed by the tweaking of numerical weights in the AOF, in the hope that successive optimization runs will ultimately converge at least to a satisfactory design. This process has been referred to as the integrated generating and choosing (IGC) approach.⁴

The often laborious and frustrating process of IGC has led to the development of the second general option for identifying optimal solutions. This option involves a two-phase approach: First generate a representative set of good designs (Pareto solutions), and second choose the most attractive design from within this set. This method has been referred to as the generate first—choose later (GFCL) approach.⁴ Unfortunately, this method entails two possible complications: 1) the designer or decision maker might find it undesirable or impractical to examine an unduly large number of Pareto solutions in order to choose the most attractive one, and 2) the evaluation of a single Pareto solution can be prohibitively expensive. In spite of the possible complications of the GFCL approach just discussed, there can be good reasons to generate a set of evenly spaced Pareto points in design space. We note that by even distribution we mean that no one part of the Pareto frontier is over- or underrepresented in the Pareto set. A more formal definition of even distribution is provided in Sec. III.

B. Survey of Pareto Set Generators

In examining approaches to generate Pareto sets, we first turn to the most commonly used method in multiobjective optimization—the weighted sum (WS) method. The AOF of the WS method is a linear combination of design objectives. This linear form has well-known deficiencies that limit its ability to effectively generate Pareto solutions. The most serious disadvantage of the WS method is its inability to capture solutions on nonconvex regions of the Pareto frontier.^{4–7} In addition, evenly varying the weights in the WS AOF does not typically result in an even distribution of Pareto solutions.

Other methods have overcome the major drawbacks of the WS method. These include the compromise programming,⁶ the ϵ -constraint,¹ the physical programming,⁴ the normal boundary intersection (NBI),⁵ and the normal constraint⁸ (NC) methods. The compromise programming (CP) method is capable of generating both convex and nonconvex Pareto frontiers.⁶ The success of this method is highly dependent on the order of the AOF used. In fact, with the appropriate order of the AOF, the CP method is capable of yielding solutions belonging to a highly concave Pareto frontier. This issue is examined in detail in Messac and Ismail-Yahaya.⁷ We note that the CP method shares a deleterious characteristic of the WS method, namely, its inability to generate well-distributed solutions along the Pareto frontier—given an even distribution of scalar weights.

The physical programming⁹ method, which is primarily intended to be used in the IGC context, has also proven useful for generating Pareto sets. A recent publication has shown that by evenly changing input preferences the physical programming method generates a set of evenly distributed Pareto solutions.⁴ The physical programming method is not limited to biobjective problems and can generate Pareto solutions on convex and nonconvex regions.

Recently, the NBI method has been introduced as one that successfully generates an even distribution of Pareto solutions on convex or nonconvex Pareto frontiers for problems of n objectives. Details of this method can be found in Das and Dennis¹⁰ and Das.^{11,12} Among other applications, the NBI method has been successfully implemented in a chemical process simulator.¹³ For the chemical process simulator, the NBI method performed favorably when compared to the WS and goal programming methods. Although significantly more effective than the WS and CP methods at generating evenly distributed solutions on the Pareto frontier, the NBI method has its own drawbacks, namely, it generates non-Pareto (dominated) and locally Pareto solutions in many practical cases. Fortunately, Pareto filtering techniques can be used to easily remove non-Pareto and locally Pareto points.¹⁴

A relatively new Pareto set generator called the NC method has also been developed.⁸ Similarly to the NBI method, the NC method generates an evenly distributed set of Pareto solutions in an n -dimensional design space, be it convex or not. By its formulation, the NC method has been shown to offer important advantages over the NBI method,⁸ namely, the NC method is more computationally stable, and is less likely to generate non-Pareto and locally Pareto solutions when compared to the NBI method. These advantages are

discussed in Sec. V, where the NC and NBI methods are compared. The similarity between the NC and NBI methods stems from both being partially based on the works of Gembicki¹⁵ and Schy and Giesy.¹⁶

Both the NBI and NC methods have a serious limitation, which is remedied in this paper for the NC method. Specifically, the process of generating Pareto sets under these methods is such that some regions of the feasible design space are left unexplored. As a result, any Pareto solution in these unexplored regions will remain unidentified. For Pareto frontier based design to be truly successful, a set of evenly distributed Pareto solutions over the complete Pareto frontier must be obtained. Under their current levels of development, both the NBI and NC methods fail to generate Pareto solutions over the complete Pareto frontier. As such, the designer is left with an incomplete understanding of the design space and an inability to choose from the full range of Pareto solutions. We parenthetically note that for biobjective problems the NBI and NC methods explore the full range of Pareto solutions. The same is unfortunately not true for problems of higher dimensions.

In the spirit of the present discussion, we make some important observations regarding bilevel programming and other preemptive methods. The bilevel programming method¹⁷ generates both Pareto and non-Pareto solutions. The Pareto genetic algorithm method presents the generation of a set of Pareto solutions.¹⁸ The Pareto solutions are obtained through a combination of Pareto-set filters, such as the fuzzy logic penalty function algorithm. The method is useful for generating Pareto solutions. However, it does not generate a set of evenly distributed solutions.

Thus far we have primarily discussed representing the Pareto frontier through discrete means. In certain applications one might need a continuous representation of the Pareto frontier. We note that an even distribution of points is desirable when one forms an analytical approximate and continuous representation of the Pareto frontier.

A recent study by Tappeta and Renaud on tradeoff analysis is closely related to approximating the Pareto frontier.¹⁹ However, they use a different approach for generating the Pareto frontier. They use first- and second-order derivatives of the objectives, and they note that reducing computational error associated with the approximation requires accurate second-order derivatives. Similarly, a recent work by Fadel and Li deals with approximating the Pareto frontier for biobjective problems.²⁰ Their method, which is limited to two objectives, is used to optimize a simple structure.

C. New Phase of Development for the NC Method

This paper offers a new phase in the development of the NC method.⁸ Specifically, the new developments make it possible for the NC method to guarantee the generation of an evenly distributed set of Pareto solutions that represents the complete Pareto frontier. Such a guarantee renders the NC method more effective than the NBI and original NC methods at generating a set of representative Pareto points. The even distribution of the generated Pareto solutions over the complete Pareto frontier can also facilitate the development of an analytical expression for the Pareto frontier in n dimension. In addition, it facilitates the task of choosing the final design because the designer can examine the entire range of candidate optimal solutions before making a decision.

The remainder of the paper is presented as follows. Section II provides technical preliminaries and definitions. A new developmental perspective on the NC method is presented in Sec. III, which is followed by important additional NC method developments in Sec. IV. In Sec. V, examples are provided that illustrate the effectiveness of the NC method under the new developments. Also in Sec. V, a discussion outlining the differences between normal constraint and normal boundary intersection methods is provided. Concluding remarks are given in Sec. VI.

II. Technical Preliminaries

This section provides technical preliminaries that facilitate the presentation of the NC method and the new developments presented in this paper. A generic multiobjective optimization (MO) problem statement is defined, followed by a description of key reference

points used in the development of the normal constraint method. Finally, we discuss solving the MO problem under a reduced feasible space, which is an important aspect of the NC method.

A. Generic Multiobjective Optimization

In practical contexts, a designer often needs to optimize disparate conflicting objectives—an activity often referred to as multiobjective optimization. The generic MO problem can be stated as shown in problem 1.

Problem 1: Generic multiobjective optimization

$$\min_x \mu(x) \quad (1)$$

Subject to:

$$g(x) \leq 0 \quad (2)$$

$$h(x) = 0 \quad (3)$$

$$x_l \leq x \leq x_u \quad (4)$$

As discussed in Sec. I, an important class of solutions to the MO problem is said to be Pareto optimal.

Associated with every MO problem is a feasible design space. By definition, a design solution within the feasible design space satisfies the constraints. Figure 1a shows a feasible design space (shaded volume) for a three-objective case. The mutually orthogonal axes represent individual design objectives. For problems of more than three objectives, the feasible design space is a hypervolume.

Also associated with every MO problem are the important reference points defined next:

Anchor points are specific designs, in the feasible design space, that correspond to the best possible values for respective individual objectives. For a biobjective problem, the anchor points are labeled as μ^{1*} and μ^{2*} in Fig. 2. The i th anchor point is written as

$$\mu^{i*} = [\mu_1(x^{i*}) \quad \mu_2(x^{i*}) \quad \dots \quad \mu_m(x^{i*})]^T \quad (5)$$

where $x^{i*} = \arg \min_x \mu_i(x)$ subject to Eqs. (2–4).

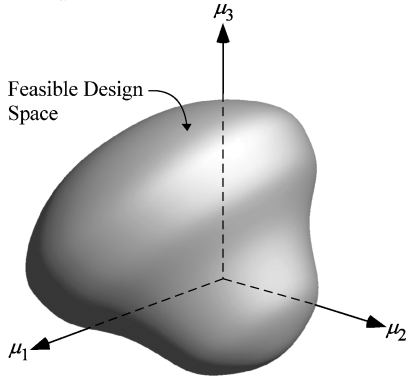


Fig. 1a Feasible design space (shaded volume) for three-objective case.

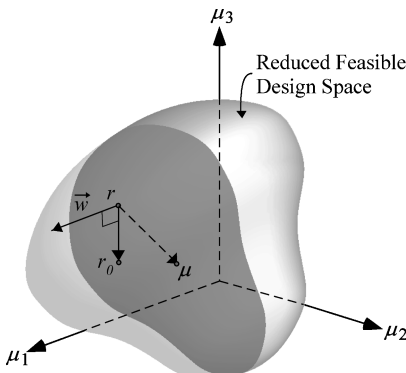


Fig. 1b Generic reduction of feasible design space.

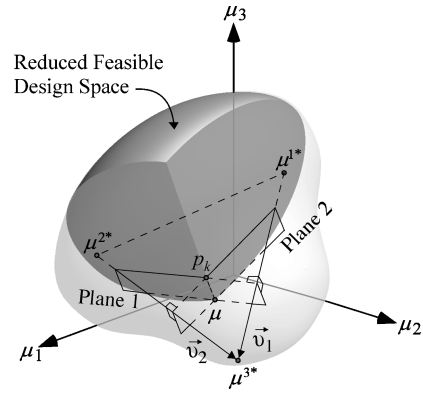


Fig. 1c Normal constraint based reduction of feasible design space.

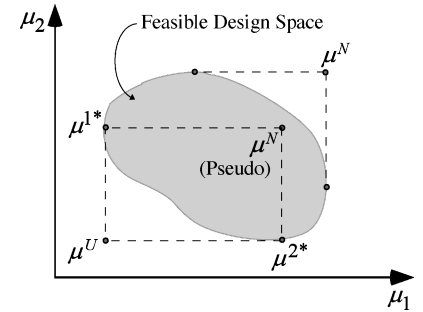


Fig. 2a Graphical description of multiobjective reference points.

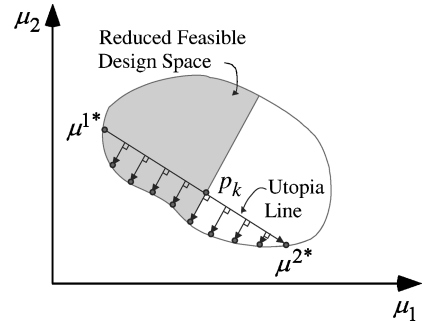


Fig. 2b Design space reduction under the NC method for a biobjective case.

Utopia point is a specific point, generally outside of the feasible design space, that corresponds to all objectives simultaneously being at their best possible values. The utopia point is denoted as μ^U in Fig. 2 and is written as

$$\mu^U = [\mu_1(x^{1*}) \quad \mu_2(x^{2*}) \quad \dots \quad \mu_m(x^{m*})]^T \quad (6)$$

Nadir point is a point in the design space where all objectives are simultaneously at their worst values. The nadir point is written as

$$\mu^N = [\mu_1^N \quad \mu_2^N \quad \dots \quad \mu_m^N]^T \quad (7)$$

where μ_i^N is defined as

$$\mu_i^N = \max_x \mu_i(x) \quad (8)$$

subject to Eqs. (2–4).

We note that another useful way to define μ_i^N is

$$\mu_i^N = \max \{ \mu_i(x^{1*}), \dots, \mu_i(x^{m*}) \} \quad (9)$$

We refer to the point defined in Eq. (7) as the *pseudo nadir point* when Eq. (9) is used to define μ_i^N . As shown in Fig. 2, the pseudo nadir point is one in the design space with the worst design objective

values of the anchor points. To guarantee full coverage of the Pareto frontier, we must use Eq. (8) to define μ_i^N . However, experience indicates that the definition of Eq. (9) is an efficient alternative, which still adds great benefit over the original NC method as well as the NBI method.

We note that the developments presented in this paper are completely valid for an arbitrary number of objectives. We will however restrict much of the discussion to the three-objective case, in order to preserve the ability to describe the approaches graphically.

B. MO Problem Under a Reduced Feasible Space

The NC method is based on a sequence of systematic design space reductions. In general, the design space will be reduced through imposing additional constraints on the MO problem, which are termed reduction constraints. Figure 1b shows a reduced design space, which is a subset of the original design space shown in Fig. 1a.

In constructing reduction constraints, recall that the equation of a plane can be defined by an arbitrary point r_0 on the plane and a vector w that is orthogonal to that plane. The vector equation of a plane is then expressed as

$$w \cdot (r_0 - r) = 0 \quad (10)$$

where r is a point on the plane. Setting the dot product equal to zero ensures orthogonality. Figure 1b shows r , r_0 , and w for defining the plane that is parallel to the μ_2 - μ_3 plane. To solve problem 1 subject to this reduced feasible space, the following constraint is added to problem 1:

$$w \cdot (\mu - r) \leq 0 \quad (11)$$

where μ is a generic point in the design space. Note that the reduced feasible design space, shown in Fig. 1b, can be described by the constraints given in Eqs. (2–4) and Eq. (11) and that any of these constraints might or might not be active in a particular problem.

III. NC Method: New Developmental Perspective

A new developmental perspective of the NC method⁸ is described in this section. Important new developments for the NC method are then introduced in Sec. IV. In Sec. III.A, a brief description of the NC method is presented. In Sec. III.B, the NC method's unique approach to reducing the feasible design space is described. Section III.C presents the basic algorithm for sequentially reducing the feasible design space and optimizing to obtain a set of Pareto solutions. The NC method's approach for handling disparate objective magnitudes (scales) is then presented in Sec. III.D. Finally, in Sec. III.E the current limitations of the NC method are discussed.

A. Introduction to NC Method

The NC method is a Pareto frontier generator that has the following characteristics. The NC method 1) generates an even distribution of Pareto points along the Pareto frontier (see definition of even distribution next), 2) is insensitive to design objective magnitude, 3) is valid for an arbitrary number of design objectives, and 4) is relatively easy to implement. Importantly, we note that under the new developments presented in Sec. IV the NC method will 1) be guaranteed to yield any Pareto point in the feasible design space and 2) generate a set of evenly distributed Pareto points along the complete Pareto frontier.

The NC method obtains a set of evenly distributed Pareto solutions for a generic MO problem (problem 1) by performing a series of optimizations. Each optimization in the series is performed subject to a reduced feasible design space. With each design space reduction, a single Pareto solution is obtained by 1) transforming the original MO problem to a single-objective problem and 2) minimizing the single-objective subject to the reduced design space. Starting with the original feasible design space and reducing it until the entire design space has been explored, the NC method generates Pareto solutions throughout the Pareto frontier. In the sense that it is based on a design space reduction scheme, the NC method is similar to other Pareto frontier generators (e.g., ϵ -constraint and NBI

methods). However, the NC method is distinct in its approach (as shown in this and the next section) and in its effectiveness (as shown in the examples). Additionally, the new developments presented in Sec. IV give the NC method the unique feature of guaranteeing full Pareto frontier coverage.

The NC method entails two critical aspects that result in the generation of an evenly distributed set of Pareto solutions. The first aspect is that of judiciously reducing the feasible design space. The second aspect is that of choosing a sequence of reductions and optimizations that result in an evenly distributed set of Pareto solutions. The design space reduction approach is presented in Sec. III.B, and the development of a sequential reduction and optimization strategy is developed in Sec. III.C.

Definition: Even distribution is a set of points that is evenly distributed over a region if no part of that region is over- or underrepresented in that set of points, compared to other parts. A measure of distribution evenness is described next.

One approach for measuring the evenness of a distribution of points can be understood through the illustration in Fig. 3. In this figure, each axis represents a design objective. To measure the evenness of the distribution of points in Fig. 3a, we construct two circles for each point μ^i in the set of points. For both circles, the circle diameter joins two points in the set. One circle is the smallest circle that can be constructed between point μ_i and any other point in the set. The diameter of this circle is denoted as d_l^i . The other circle is constructed so that its diameter d_u^i is the maximum distance from point μ^i to any other point in the set such that no point in the set is within the circle. A measure of the evenness is given by the expression $\xi = \sigma_d / \hat{d}$, where \hat{d} and σ_d denote the mean and standard deviation of d , respectively, and where $d^i = \{d_l^i, d_u^i\}$ and $d = \{d^1, \dots, d^{2np}\}$. A set of points is exactly evenly distributed when $\xi = 0$. We assume that the objective space is normalized such that $0 \leq \mu_i \leq 1, \forall i \in \{1, 2, \dots, m\}$.

For the cases of n dimensions (see Fig. 3b), the diameter of a hypersphere is considered in place of the circle shown in Fig. 3a. The right side of Fig. 3b shows that points are (locally) evenly distributed when the diameters of the spheres are nearly equal and that points are not evenly distributed (left-hand side of figure) when the diameters of the spheres are not close in value.

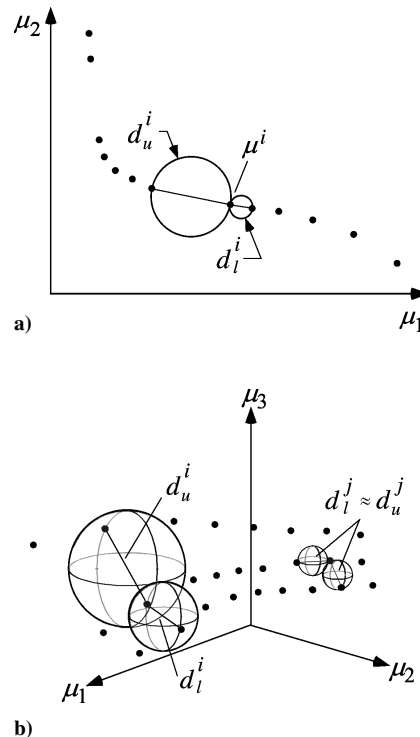


Fig. 3 Graphical description of factors contributing to the evenness of a distribution of points.

B. Judicious Reduction of Feasible Design Space

One of the main distinguishing factors between the NC and other methods is the approach used for reducing the feasible design space. Under the NC method, a so-called utopia plane plays a critical role in this reduction (see the following definition of utopia plane).

Definition: Utopia plane is a hyperplane constructed such that it contains all anchor points. Note that the utopia plane does not contain the utopia point.

As defined, the utopia plane has an important relation to the Pareto frontier. Specifically, an evenly distributed set of points on the utopia plane can be used to define reduction constraints that lead to evenly distributed points on the Pareto frontier. An even distribution of points on the Pareto frontier occurs when reduction constraints are made normal to the utopia plane. Figure 2b shows that the reduction constraints, which are normal to the utopia plane (utopia line), result in an even distribution of Pareto solutions for a biobjective case. This important characteristic is what makes the NC method effective at generating even distributions of points on the Pareto frontier. From Fig. 2b, it can be seen that obtaining an even distribution requires similar scales for the design objectives. Importantly, Sec. III.D provides a normalization approach that overcomes scaling problems and ensures obtaining a set of Pareto solutions that represents the Pareto frontier well. In this section, the development of NC based design space reduction is presented, and the next section examines a sequence of reductions and optimizations.

Figure 1c shows the NC-based approach to design space reduction. The solid volume with two flat surfaces represents the reduced feasible design space. A section of the utopia plane is shown as a triangle with the anchor points at the vertices. Planes 1 and 2 define the flat surfaces of the reduced feasible design space and correspond to optimization constraints that make all but the shaded volume infeasible. Note that this reduction approach is distinct from the generic approach shown in Fig. 1b and is a key contributor in the NC method's ability to generate evenly distributed Pareto points. Planes 1 and 2, and the corresponding optimization constraints, are obtained using the NC feasible space reduction process (NCSR) process as shown here:

Step 1: Compute $m - 1$ vectors. One vector from the i th anchor point to the j th anchor point for all $i \neq j$, where j is an arbitrarily given design objective,

$$v_i = \mu^{j*} - \mu^{i*}, \quad \forall i \in \{1, \dots, m\}, \quad i \neq j \quad (12)$$

Step 2: Choose a generic point p_k on the utopia plane. (The utopia plane space is parameterized in Sec. III.C.)

Step 3: Reduce the feasible space by enforcing the following set of plane constraints:

$$v_i \cdot (\mu - p_k) \leq 0, \quad \forall i \in \{1, \dots, m\}, \quad i \neq j \quad (13)$$

where μ is a generic point in the feasible design space. When Eq. (13) is equal to zero, it represents the equation of a plane that is perpendicular to the utopia plane containing point p_k .

The reduced feasible design space shown in Fig. 1c can be described by the set of optimization constraints

$$g(x) \leq 0 \quad (14)$$

$$h(x) = 0 \quad (15)$$

$$x_l \leq x \leq x_u \quad (16)$$

$$v_i \cdot (\mu - p_k) \leq 0, \quad i = 1, 2 \quad (17)$$

$$v_i = \mu^{3*} - \mu^{i*}, \quad i = 1, 2 \quad (18)$$

The NCSR process reduces the design space for a generic point p_k on the utopia plane. Section III.C below describes how p is obtained and used to perform a series of optimizations that yield a set of evenly distributed Pareto solutions.

C. Sequential Reduction and Optimization

Under the NC method, the NCSR process (developed in Sec. III.B) is carried out for a set of evenly distributed points p_k on the utopia plane. By so doing, the feasible design space is systematically and sequentially reduced. With each reduction, a single-objective optimization problem is solved, which results in a single Pareto solution for the original multiobjective problem.

In presenting the approach for generating evenly distributed points on the utopia plane, we again consider a three-objective case as shown in Fig. 4a. Here, a section of the utopia plane is shown as a triangle with the anchor points at the vertices. A set of evenly distributed points is shown on the utopia plane. Any point on the utopia plane can be defined as a function of the anchor points. In general, the i th point on the polygon formed by the anchor points (triangular section of the utopia plane for three objective case) can be written as

$$p_i = \sum_{j=1}^m \alpha_i^j \mu^{j*} \quad (19)$$

where the nondimensional parameter α_i^j satisfies

$$0 \leq \alpha_i^j \leq 1 \quad (20)$$

$$\sum_{j=1}^m \alpha_i^j = 1 \quad (21)$$

By varying α^j from 0 to 1 with a fixed increment of δ^j , an even distribution of points on the utopia plane can be generated. Figure 5 shows the values of α^j for a three-objective case, where $\delta^j = 0.2$ for all α^j .

To generate a set of evenly distributed points along the Pareto frontier, we reduce the design space using the NCSR process for each point p_k obtained using Eqs. (19–21), where α^j is evenly varied

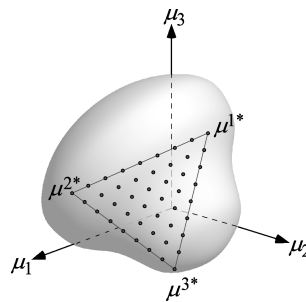


Fig. 4a Set of evenly distributed points on the utopia plane.

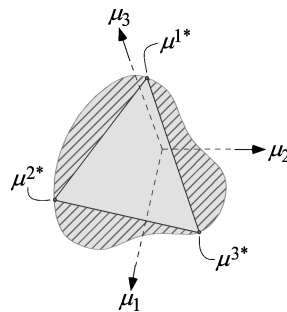


Fig. 4b Unobtainable Pareto solutions (hatched regions) under the NBI and originally developed NC methods. (View is rotated so as to be normal to utopia plane.)

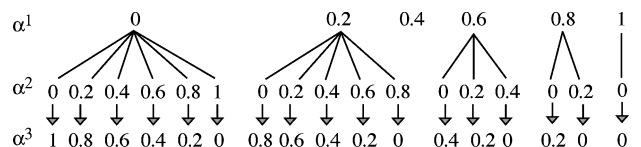


Fig. 5 Nondimensional parameters used in the generation of points on the utopia plane.

from 0 to 1. Importantly, the MO problem (problem 1) is transformed to a single-objective optimization problem as shown in problem 2. problem 2 is solved for each point p_k on the utopia plane. As such, problem 2 is a generic optimization problem of the normal constraint method.

Problem 2a: Normal constraint generic optimization problem for point p_k :

$$\min_x \mu_j(x) \quad (22)$$

Subject to:

$$g(x) \leq 0 \quad (23)$$

$$h(x) = 0 \quad (24)$$

$$x_l \leq x \leq x_u \quad (25)$$

$$v_i \cdot (\mu - p_k) \leq 0, \quad \forall i \in \{1, \dots, m\}, \quad i \neq j \quad (26)$$

$$v_i = \mu^{j*} - \mu^{i*}, \quad \forall i \in \{1, \dots, m\}, \quad i \neq j \quad (27)$$

where $j \in \{1, \dots, m\}$. Note that the reduction constraint obtained from the NCSR process is included in the optimization problem statement. Solving problem 2a for a set of evenly distributed points p_k will result in an evenly distributed set of Pareto solutions—when the objectives ranges are of similar magnitudes. When they are not of similar magnitudes, the development of Sec. III.D can be used.

D. Generating Even Distributions Under Disparate Objectives Scales

If the design objectives in problem 1 have different magnitudes, such as the volume of a structure vs its deformation (when exposed to loads), then the objectives must first be normalized in order to obtain a set of Pareto solutions that represents the Pareto frontier well. The objectives can be normalized by the following approach.

The normalized value of the design objective (denoted as $\bar{\mu}$) can be computed using the utopia point and pseudo nadir point (see Sec. II). The following equation is used to perform the mapping:

$$\bar{\mu}_i = \frac{\mu_i - \mu_i^U}{\mu_i^N - \mu_i^U} \quad (28)$$

Note that the normalization is performed in the design objective space—not the design variable space. The normalized form of problem 2a is stated as problem 2b.

Problem 2b: Normalized normal constraint generic optimization problem for point p_k :

$$\min_x \bar{\mu}_j(x) \quad (29)$$

Subject to:

$$g(x) \leq 0 \quad (30)$$

$$h(x) = 0 \quad (31)$$

$$x_l \leq x \leq x_u \quad (32)$$

$$\bar{v}_i \cdot (\bar{\mu} - \bar{p}_k) \leq 0, \quad \forall i \in \{1, \dots, m\}, \quad i \neq j \quad (33)$$

$$\bar{v}_i = \bar{\mu}^{j*} - \bar{\mu}^{i*}, \quad \forall i \in \{1, \dots, m\}, \quad i \neq j \quad (34)$$

where $j \in \{1, \dots, m\}$. Performed repeatedly for a set of evenly distributed points p_k , problem 2b will yield an even distribution in the normalized space. This distribution in the normalized space directly leads to a set of Pareto solutions that represents the objectives ranges in the nonnormalized space well.

Thus far in Sec. III, we have described the basic NC approach for generating Pareto solutions that are evenly distributed along the Pareto frontier. We have shown that the key to the NC method is the procedure for reducing the design space and the approach for sequential reduction and optimization. In a recent publication, the

NC method as just described has been shown to perform favorably when compared to the WS, CP, and NBI methods.⁸ Section III.E discusses a significant limitation of the NC method, which is also shared by similar methods, such as the NBI method.

E. Limitations of the Current NC Method

Under the NC method just presented, there is no guarantee that the generated set will represent the complete Pareto frontier for problems of more than two objectives. More specifically, the NC method leaves some regions of the Pareto frontier unexplored (i.e., ignored). The NBI method also suffers from this limitation. For example, when the points generated on the utopia plane section in Fig. 4a are used in the NCSR process, the hatched regions shown in Fig. 4b are left unexplored. Note that the design space in Fig. 4b has been rotated so as to present a view that is normal to the utopia plane. This discussion leads us to a fatal flaw of the NC and NBI methods, namely, that the hatched region of Fig. 4b is generally unobtainable. In a publication introducing the NBI method, Das and Dennis¹⁰ state that these unobtained Pareto solutions are not as interesting from the tradeoff point of view.

We make the important observation that because any Pareto solution can potentially be the most preferred by the designer it is critical to be able to generate any of these points. Stated differently, any two Pareto points are objectively of equal value; only the subjective preference of the designer makes one more desirable than the other. As such, none should be unreachable. Mattson et al.¹⁴ propose the notion of practically insignificant tradeoff between neighboring Pareto points, which provides for a quantitative means for the designer to reduce the number of Pareto solutions under consideration. However, even under that approach, no section of the Pareto frontier is entirely ignored, as the hatched region of Fig. 4b warns us.

The new developments, presented in Sec. IV, overcome the limitations of the current NC method. As such, the NC method acquires the important (and unique) ability to guarantee that it generates a set of evenly distributed Pareto solutions over the complete Pareto frontier.

IV. Normal Constraint Method: Representing the Complete Pareto Frontier

This section provides important developments that overcome the limitations of the originally developed NC method. Specifically, the new developments ensure that the generated Pareto set represents the complete Pareto frontier. In the discrete domain, it allows us to generate the entire Pareto frontier, which is a powerful feature that the originally developed NC and NBI methods do not possess. The NC-based procedure for generating the complete frontier entails three main steps, which are briefly described next. The development of each step is then presented in the sequel:

Step A: Create a hypercube that encloses the entire feasible space. To generate the complete Pareto frontier, we first identify a volume in which all Pareto solutions must lie. Such a volume is illustrated in Fig. 6a. This hypercube is constructed such that the utopia and nadir points occupy opposite corners. By definition of the utopia and nadir points, all feasible designs are enclosed by this hypercube. Therefore, we can guarantee that the complete Pareto frontier is generated by ensuring that the space within the hypercube is thoroughly explored.

Step B: Size the utopia plane section. The hypercube obtained in step A can be used together with the NC method to ensure that all Pareto solutions are generated. Consider the design objective space shown in Fig. 7. The space has been rotated to present a view that is normal to the utopia plane. The small triangle represents the utopia plane section within which points p were generated and used under the originally developed NC method. The large triangle represents an enlarged utopia plane section used under the new developments. Specifically, under this step, we enlarge the utopia plane section until it encloses the normal projection of the hypercube onto the utopia plane. Doing this will ensure that Pareto points anywhere in the hypercube can be obtained under the NC method.

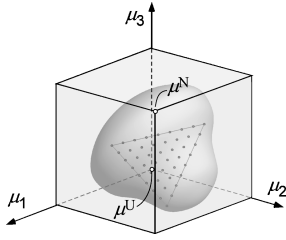


Fig. 6a Hypervolume that encloses entire feasible space.

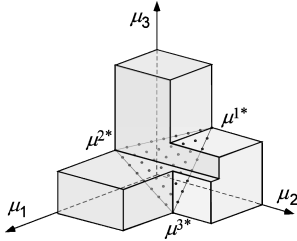
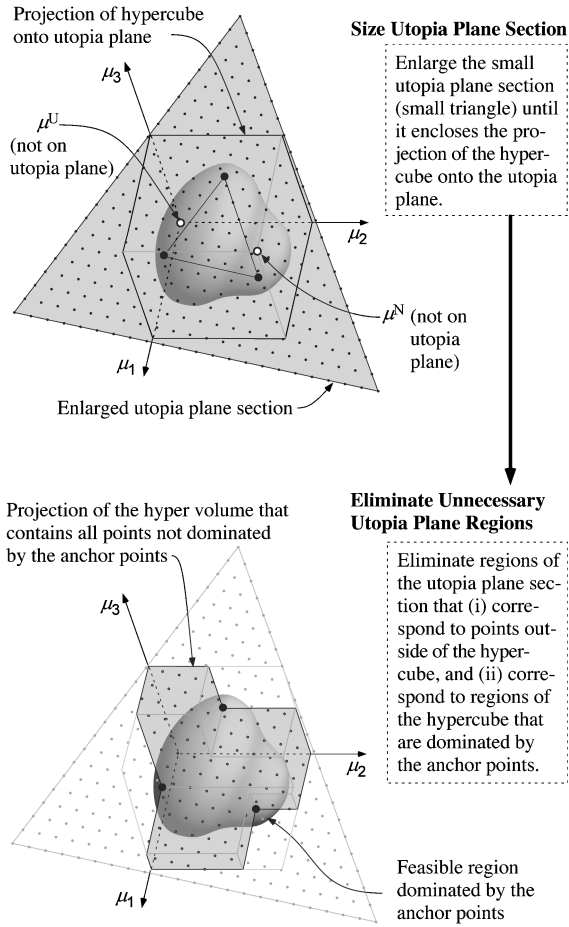


Fig. 6b Design objective space that is not dominated by the anchor points $(\mu^{1*}, \mu^{2*}, \mu^{3*})$.



LEGEND

- Anchor point
- Point p_k on utopia plane
- Utopia plane section
- Feasible space

Fig. 7 New approach for generating points p on the utopia plane.

To resize the utopia plane section, we let the parameterization constraint given in Eq. (20) become $\alpha_i^j \leq \alpha_i^j \leq \alpha_u^j$, where α_i^j and α_u^j are yet to be determined. To find the optimal size of the utopia plane section, we solve a set of computationally benign optimization problems to find α_i^j and α_u^j , subject to conditions that enlarge the section only until it completely encloses the hypercube projection. We also capitalize on known information about the utopia, nadir, and anchor points to eliminate regions of the enlarged utopia plane

section that cannot yield Pareto solutions. The lower plot in Fig. 7. shows that unnecessary regions of the utopia plane section have been eliminated. Importantly, the shaded region of the lower plot illustrates the minimal utopia plane section needed to generate the complete Pareto frontier.

The outcome of this step is a set of evenly distributed points on the utopia plane section shown in the bottom of Fig. 7. These are the only points needed to generate the complete Pareto frontier under the NC method.

Step C: Generate the complete Pareto frontier. To generate the complete Pareto frontier, the NC generic optimization problem (problem 2) is solved using the set of utopia plane points generated in step B. In rare cases, it is possible that some non-Pareto or locally Pareto solutions will be generated. To eliminate these spurious solutions, we simply use a Pareto filter that removes all but the globally Pareto solutions. The end result is a set of evenly distributed Pareto solutions that represents the complete Pareto frontier.

Each of these steps is developed next.

A. Create a Hypercube Enclosing Entire Feasible Space

A volume of the design space that comprises all Pareto solutions can be defined by the utopia and nadir points. As depicted in Fig. 6a, a hypercube with the utopia and nadir points in opposite corners defines such a volume. When the entire hypercube is explored, then we can be assured that all Pareto solutions are obtained. Specifically, the hypercube enclosing (at least) the entire feasible space can be defined as one comprising all designs μ where

$$\mu_i^U \leq \mu_i \leq \mu_i^N, \quad \forall i \in \{1, \dots, m\} \quad (35)$$

The hypercube can be projected (perpendicularly) to the utopia plane where it is shown as a hexagon in Fig. 7. Importantly, this projection plays a critical role in generating the complete Pareto frontier. Specifically, the volumetric prism extending perpendicularly to the hexagon encloses all possible Pareto points and represents the region explored under the NC method.

If the small triangular utopia plane section (Fig. 7) can be enlarged to enclose the projection of the hypercube (hexagon), then all Pareto solutions will be generated under the NC method.

B. Size the Useful Utopia Plane Section

Under the second step in the process of generating the complete Pareto frontier, the utopia plane section is sized so that it encloses the projection of the hypercube developed in Sec. IV.A. The sizing occurs by allowing the parameterization of the utopia plane space to be modified from that of the originally developed NC method (Sec. III). Recall that under the parameterization of Sec. III the parameter α_i^j was restricted as $0 \leq \alpha_i^j \leq 1$ and that

$$\sum_{j=1}^m \alpha_i^j$$

was forced to be equal to one. The former restriction keeps the generated point p_i inside the polygon formed by the anchor points, while the latter keeps the point p_i on the utopia plane. Under the developments of this section, α_i^j is not constrained to be between zero and one, thus allowing the parameterization to extend beyond the polygon formed by the anchor points. The summation of α_i^j is still confined to be equal to one so that all of the generated points p are on the utopia plane. Therefore, the i th point on the enlarged utopia plane section can be written as follows:

$$p_i = \sum_{j=1}^m \alpha_i^j \mu^{j*} \quad (36)$$

where the nondimensional parameter α_i^j satisfies

$$\alpha_l^j \leq \alpha_i^j \leq \alpha_u^j \quad (37)$$

$$\sum_{j=1}^m \alpha_i^j = 1 \quad (38)$$

By varying α^j from α_l^j to α_u^j with a fixed increment of δ^j , an even distribution of points on the enlarged utopia plane section can be generated.

To find the limits for enlarging the utopia plane section, we obtain the upper and lower limits of the parameter α^j by solving a set of computationally benign optimization problems written as problems 3a and 3b.

Problem 3a: Computationally benign optimizations for obtaining lower limit of α^j , where $j \in \{1, \dots, m\}$:

$$\alpha_l^j = \min_{\mu, \alpha} \alpha^j \quad (39)$$

Subject to:

$$v_k \cdot (\mu - p) = 0, \quad \forall k \in \{1, \dots, m\}, \quad k \neq r \quad (40)$$

$$v_k = \mu^{r*} - \mu^{k*}, \quad \forall k \in \{1, \dots, m\}, \quad k \neq r \quad (41)$$

$$p = \sum_{i=1}^m \alpha^i \mu^{i*} \quad (42)$$

$$\sum_{i=1}^m \alpha^i = 1 \quad (43)$$

$$-\infty \leq \alpha^i \leq \infty, \quad \forall i \in \{1, \dots, m\} \quad (44)$$

$$\mu^U \leq \mu \leq \mu^N \quad (45)$$

where $r \in \{1, \dots, m\}$ and μ is a generic point in the hypervolume constrained by Eq. (35). To make the problem computationally benign, μ has been made a dummy variable that must satisfy the conditions of Eq. (45) and is not dependent on x as in the actual multiobjective optimization problem.

We now make an important note about the constraint given in Eq. (40). This constraint capitalizes on the fact that a vector normal to the utopia plane can be constructed through every Pareto point. As such, this constraint verifies that the utopia plane section undergoes a minimum enlargement such that a vector can be constructed perpendicularly to the utopia plane section from each point μ in the hypercube. Simply stated, this constraint allows the utopia plane section to be enlarged only until it completely encloses the projection of the hypercube.

Similarly, to find the upper limit on the parameter α^j , the following optimization can be carried out.

Problem 3b: Computationally benign optimization for obtaining upper limit of α^j , where $j \in \{1, \dots, m\}$:

$$\alpha_u^j = \max_{\mu, \alpha} \alpha^j \quad (46)$$

subject to Eqs. (40–45). Problem 3 is carried out for $\alpha^j \forall j \in \{1, \dots, m\}$.

Having obtained the upper and lower limits of the parameter α^j , we can now generate a set of evenly distributed points on the utopia plane by varying α_l^j from α_l^j to α_u^j by even increments. Such a generation, yields a set of points in the space represented by the large triangle of Fig. 7.

Although the enlarged utopia plane section, just discussed, is guaranteed to result in the exploration of the complete Pareto frontier, it also unfortunately explores large regions of the space where no feasible solutions exist. To prevent useless searches and to make the process more efficient, computationally benign feasibility tests (problem 4) can be performed to identify points on the utopia plane from which normal vectors can be constructed to lead to at least one potentially feasible, nondominated, design point. The lower plot in Fig. 7 depicts the utopia plane section resulting after unnecessary regions are removed. To guarantee that the complete Pareto frontier will be represented, only the points in the reduced utopia plane sections are needed when carrying out the NC method.

Two particular conditions lead to the ability to eliminate unnecessary regions of the utopia plane section—and still guarantee that the complete Pareto frontier is generated. They are 1) the condition that no Pareto point is better or worse than the utopia and nadir points, respectively, and 2) the condition that no Pareto point is dominated by any of the anchor points. Specifically, Eq. (50) ensures that we only search for Pareto solutions in regions that are not dominated by the anchor points. We perform the following computationally benign feasibility tests to remove points from the utopia plane section that violate these conditions:

Problem 4: Computationally benign check of point p_i usefulness:

$$\min_{\mu} C \quad (47)$$

Subject to:

$$v_k \cdot (\mu - p_i) = 0, \quad \forall k \in \{1, \dots, m\}, \quad k \neq j \quad (48)$$

$$v_k = \mu^{j*} - \mu^{k*}, \quad \forall k \in \{1, \dots, m\}, \quad k \neq j \quad (49)$$

$$\sum_{i=1}^m [\mu_i - \mu_i(x^{r*})] \leq \sum_{i=1}^m |\mu_i - \mu_i(x^{r*})| - \epsilon \quad (50)$$

$$\mu_i^U \leq \mu_i \leq \mu_i^N, \quad \forall i \in \{1, \dots, m\} \quad (51)$$

where r and $j \in \{1, \dots, m\}$, ϵ is a small number, and C is a constant. All points (p_i) that satisfy problem 4 are retained as candidates for exploring the useful design space. Importantly, problem 4 is repeated for each anchor point (i.e., $\forall r \in \{1, \dots, m\}$). The result of step B is a set of evenly distributed points on the shaded region of the utopia plane section, shown in the lower part of Fig. 7.

C. Generate the Complete Pareto Frontier

To generate the complete Pareto frontier, the NC generic optimization problem (problem 2) is solved using each point in the set resulting from step B. Importantly, we note that only the points obtained from step B are needed to generate the complete Pareto frontier. As discussed in Sec. I, there are some special cases where the NC method results in locally Pareto or non-Pareto solutions (in addition to all of the Pareto points). To overcome this deficiency, a simple Pareto filter can be used.^{8,14} Pareto filters examine a set of candidate solutions and remove all that are not globally Pareto optimal.

Under the developments presented in this section, the NC method is guaranteed to generate a set of Pareto points that evenly represents the complete Pareto frontier, whereas in Sec. III only an even representation can be guaranteed. In the event that the designer finds the computational effort of the developments of this section is not warranted, the operations described in Sec. III can be performed alone—with the understanding that the complete Pareto frontier is not likely to be obtained. Again, for biobjective problems the developments in Sec. III are sufficient to ensure an even representation of the complete Pareto frontier.

In summary, we have shown that the limitations of the NC method as described in Sec. III can be overcome by modifying the approach used to generate the points p on the utopia plane. The method described in Sec. III often leaves Pareto frontier regions unobtainable and therefore unidentified. The developments presented in this section provide a new method for generating the points p on the utopia plane. The new method ensures that all Pareto regions can be explored, while still maintaining an efficient search.

V. Numerical Examples

In this section, we consider two examples that show the effectiveness of the NC method for generating Pareto frontiers. The first example, which is purely mathematical, illustrates one of the main differences between the NC and NBI methods. Additional details regarding the differences between the NC and NBI methods are also presented as part of the discussion. The second example entails the optimization of a simple structure and demonstrates the importance

of the new developments presented in this paper. Our examples are limited to two and three objectives in order to present the results graphically.

A. Example 1: Comparison of NC and NBI Methods

This example illustrates one of the main differences between the NBI and NC methods as it relates to the generation of non-Pareto solutions. The following biobjective problem is solved under the NC and NBI methods. Results are provided for both while only NC formulation is given:

$$\min_x [\mu_1 \quad \mu_2]^T \quad (52)$$

Subject to:

$$5e^{-x_1} + 2e^{-0.5(x_1-3)^2} - x_2 \leq 0 \quad (53)$$

$$\mu_1 = x_1 \quad (54)$$

$$\mu_2 = x_2 \quad (55)$$

$$0 \leq x_i \leq 5, \quad i = 1, 2 \quad (56)$$

The preceding problem is converted to the form of problem 2, where μ_2 is arbitrarily chosen as the objective to be minimized (similar results would be obtained if μ_1 were chosen):

$$\min_x \bar{\mu}_2 \quad (57)$$

Subject to:

$$5e^{-x_1} + 2e^{-0.5(x_1-3)^2} - x_2 \leq 0 \quad (58)$$

$$\mu_1 = x_1 \quad (59)$$

$$\mu_2 = x_2 \quad (60)$$

$$\bar{v}_1 \cdot (\bar{\mu} - \bar{p}_i) \leq 0 \quad (61)$$

$$\bar{v}_1 = (\bar{\mu}^{2*} - \bar{\mu}^{1*}) \quad (62)$$

$$0 \leq x_i \leq 5, \quad i = 1, 2 \quad (63)$$

where

$$\bar{p}_i = \sum_{j=1}^2 \alpha_j^i \bar{\mu}^{j*}$$

and α^j is varied from zero to one by increments of 1/29.

The preceding problem is solved 30 times, once for each point \bar{p}_i . Figure 8 shows the resulting Pareto frontier. It can be seen from the plot that both the NC and the NBI methods generate an evenly distributed set of points along the Pareto frontier. It can also be seen that the NBI method generated some non-Pareto and locally Pareto solutions. The NC method is not immune from generating non-Pareto points. The rare cases where the NC method generates non-Pareto and locally Pareto solutions are discussed in Messac et al.⁸

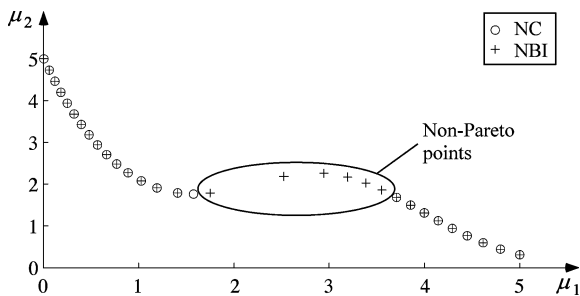


Fig. 8 Example 1 results: NC compared to NBI.

At this point, it is worth emphasizing the notable differences between the NC and NBI methods. For comprehensive information on the NBI method, see Das and Dennis.¹⁰ Three observations can be made regarding the main differences between the NC and NBI methods. They are as follows:

1) Under the new developments presented in this paper, the NC method is guaranteed to generate a set of Pareto points that represent the complete Pareto frontier, whereas the NBI method is not.

2) The NC method reduces the design space using an inequality constraint, whereas the NBI formulation uses an equality constraint, the former being computationally advantageous.

3) The NC method constructs reduction constraints based on the true normal to the utopia plane—after normalization—whereas the NBI method uses a quasi normal; here again the scaling of the objectives is favorable in the former.

Each of these three points is briefly discussed in the following after the generic NBI formulation is provided.

Problem 5: NBI problem formulation:

$$\max_{x,t} t \quad (64)$$

Subject to:

$$\Phi\beta + t\hat{n} = \mu(x) \quad (65)$$

$$g(x) \leq 0 \quad (66)$$

$$h(x) = 0 \quad (67)$$

$$x_l \leq x \leq x_u \quad (68)$$

where \hat{n} and μ are the quasi-normal vector and the vector of design metrics, respectively, and

$$\Phi = [\phi_1, \dots, \phi_m] \quad (69)$$

where $\phi_i = \mu(x^{i*}) - \mu^U$.

Generating the complete Pareto frontier is not generally possible under the NBI method. As mentioned in Sec. III, some researchers view the unobtained Pareto regions as less interesting from a trade-off perspective. We note that no one Pareto solution is objectively superior to another. Therefore it is extremely valuable to be able to generate all Pareto solutions. Under the developments of this paper, the NC method is guaranteed to generate the complete Pareto frontier.

Under the NBI method, the constraint given in Eq. (65) forces the solution to lie on the normal line \hat{n} . When the normal line does not cross the Pareto frontier, which is not uncommon, the NBI method generates a non-Pareto or locally Pareto solution. The NC method avoids this deleterious characteristic by enforcing reduction constraints of the inequality type [see Eq. (33)]. This difference causes the NBI method to generate non-Pareto solutions in example 1, whereas the NC method does not. Additionally, the use of inequality constraints is computationally favorable when compared to equality constraints.

Finally, the NBI and NC methods differ in their approaches for generating evenly distributed Pareto solutions. The NBI method uses a quasi-normal vector, whose direction equals a vector constructed between the midpoint of the utopia plane and the utopia point. A family of vectors in this direction is then used to obtain a set of Pareto solutions. An even distribution is obtained because the nature of the quasi-normal vector is such that it accounts for the disparate scales in the objectives. The NC method takes a more computationally friendly approach—a family of true normal vectors is used—in a normalized space to generate the even distribution of Pareto solutions.

B. Example 2: Representing the Complete Pareto Frontier

This example illustrates the significant impact of the developments presented in this paper. Specifically, it shows that the originally developed NC method (and NBI method) fails to generate

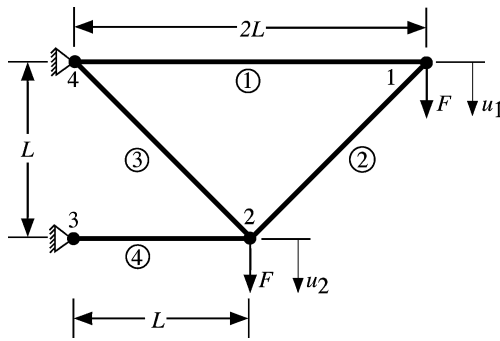


Fig. 9 Simple four-bar truss for example 2.

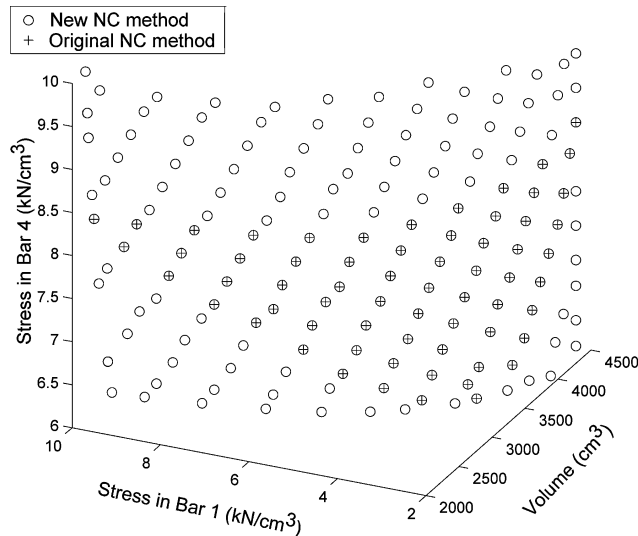


Fig. 10 Results for example 2.

large portions of the Pareto frontier that are otherwise obtained by the NC method under the developments of this paper.

A classic optimization problem²¹ is solved as part of this example—the optimization of the four-bar truss shown in Fig. 9. The truss structure is subjected to suspended loads of 10 kN each at nodes 1 and 2 as shown in the figure. As design constraints, the size of each bar must not exceed 5 cm², and the stresses in the bars are limited to 10 kN/cm² for tension and compression. The modulus of elasticity for the truss material is 2×10^4 kN/cm².

In this example, we wish to design a four-bar truss with minimal stress in bars 1 and 4. We are also interested in minimizing the volume of the structure. Specifically, the design metrics (all of which are minimized) are 1) the stress in bar 1 μ_1 , 2) the stress in bar 4 μ_2 , and 3) the volume of the structure μ_3 .

The problem is solved using the NC method in its original form and under the new developments presented in this paper. Figure 10 shows the resulting Pareto surface. The solutions obtained under the original NC method are shown as crosses, whereas the solutions obtained under the new developments are shown as circles. Clearly, the original method failed to identify large portions of the Pareto frontier. The NC method under the new development captures these regions. Also note that the Pareto solutions are generally evenly distributed, thus providing a good representation of the complete Pareto frontier. From a practical perspective, the new developments have significantly strengthened the NC method, in that it is now capable of providing the designer with the full range of Pareto solutions.

We note that the pseudo nadir point was used in this example, and that under the NC method additional Pareto points are typically generated near the extreme boundaries of the Pareto frontier. Because these additional points were already represented, they were not redundantly plotted. The generation of these additional points is because of the reduction constraint, which is of the inequality

constraint type. Under the NBI method, where the analog of the reduction constraint is of the equality type, a series of non-Pareto points is generally generated in place of the NC method's duplication of some Pareto solutions.

VI. Conclusions

An important phase in the development of the normal constraint method was presented in this paper. The new developments ensure that the normal constraint method generates a set of evenly distributed Pareto solutions that represent the complete Pareto frontier. Under the new developments the normal constraint method bears the following characteristics: 1) it generates an even distribution of Pareto points throughout the complete Pareto frontier, 2) it is guaranteed to yield any Pareto point in the feasible design space, 3) it is insensitive to design objective scaling, 4) it is valid for an arbitrary number of design objectives, and 5) it is relatively easy to implement. The NC method was compared to other Pareto frontier generators, including the normal boundary intersection method and was shown to perform favorably. Importantly, the normal constraint method is capable of exploring the entire feasible design space, which is notably different than comparable methods.

Acknowledgments

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Orbital Mechanics, Third Edition

Vladimir A. Chobotov • The Aerospace Corporation



Designed to be used as a graduate student textbook and a ready reference for the busy professional, this third edition of *Orbital Mechanics* is structured to allow you to look up the things you need to know. This edition includes more recent developments in space exploration (e.g. Galileo, Cassini, Mars Odyssey missions). Also, the chapter on space debris was rewritten to reflect new developments in that area.

The well-organized chapters cover every basic aspect of orbital mechanics, from celestial relationships to the problems of space debris. The book is clearly written in language familiar to aerospace professionals and graduate students, with all of the equations, diagrams, and graphs you would like to have close at hand.

An updated software package on CD-ROM includes: HW Solutions, which presents a range of viewpoints and guidelines for solving selected problems in the text; Orbital Calculator, which provides an interactive environment for the generation of Keplerian orbits, orbital transfer maneuvers, and animation of ellipses, hyperbolas, and interplanetary orbits; and Orbital Mechanics Solutions.

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American Institute of Aeronautics and Astronautics
Publications Customer Service, P.O. Box 960, Herndon, VA 20172-0960
Fax: 703/661-1501 • Phone: 800/682-2422 • E-Mail: warehouse@aiaa.org
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